Assessing Analytic Geometry Understanding: Van Hiele, SOLO, and Beyond

Michael J. Bossé
Appalachian State University
bossemj@appstate.edu

Anass Bayaga
Nelson Mandela University
anass.bayaga@mandela.ac.za

Kathleen Lynch-Davis
Texas A&M University Corpus Christi
kathleen.lynch-davis@tamucc.edu

Ashley M. DeMarte
Appalachian State University
ashleydemarte97@gmail.com

In the context of an analytical geometry, this study considers the mathematical understanding and activity of seven students analyzed simultaneously through two knowledge frameworks: (1) the Van Hiele levels (Van Hiele, 1986, 1999) and register and domain knowledge (Hibert, 1988); and (2) three action frameworks: the SOLO taxonomy (Biggs, 1999; Biggs & Collis, 1982); syntactic and semantic elaborations (Kaput, 1987a, 1987b, 1989); and isomorphic, transcendent, and mixed connections (Adu-Gyamfi, Bossé, & Lynch-Davis, 2019). Along with producing a fuller analysis of student work and communication, the study found that for only the students with the lowest and highest scores regarding either their understanding or actions on the analytic geometry task might there be a predictive association between knowledge and action levels. For other students, a predictive association could not be determined. This may mean that the level of understanding a student possesses regarding a particular mathematical concept may not parallel the level of actions they use when working with an associated task.

Introduction

Through application of the Van Hiele levels of geometric understanding (Van Hiele, 1986, 1999), much meaningful research has been conducted investigating student geometric understanding (e.g., Abdullah & Zakaria, 2013; Hansen, Drews, & Dudgeon, 2014; Lim, 2011; Lynn, 2010; Shtulman & Valcarcel 2012; Vosniadou, 2013; Vosniadou et al., 2015). In a parallel manner, the Structure of the Observed Learning Outcome (SOLO) taxonomy (Biggs, 1999; Biggs & Collis, 1982) has been applied to investigate student understanding in contexts demonstrating increasing complexity in general, and in algebra in particular. These frameworks have taken prominent positions in assessing levels of student understanding.

In the authors' opinions, in many state's standards for high school mathematics, geometry seems to have been given short shrift and unceremoniously replaced with analytic algebra. With analytic geometry existing at the intersection of geometry and algebra, it must be wondered whether the Van Hiele levels, the SOLO taxonomy, or another framework is best fit to investigate student understanding in analytic geometry.

This investigation considers two lengthy transcripts involving students collaboratively wrestling with and discussing analytic geometry problems. These transcripts are analyzed using both the Van Hiele levels and the SOLO taxonomy, comparisons are made across the taxonomies, and additional frameworks are considered. Altogether, the goal of this study is to fill in gaps in the literature regarding student understanding of analytic geometry.

Background

While the Van Hiele levels (Van Hiele, 1986, 1999) have long been used to investigate student understanding in geometry and the SOLO taxonomy (Biggs, 1999; Biggs & Collis, 1982) has been a lens through which student algebraic activity has been viewed, less has been done in the context of analytic geometry – the intersection of geometry and algebra. However, attempting to investigate student work in analytic geometry employing the Van Hiele levels and the SOLO taxonomy immediately introduces novel dynamics. First, it must be recognized that the two frameworks differ in context and purpose. The Van Hiele levels consider a student's level of understanding and the SOLO taxonomy considers the actions a student takes when interacting with increasingly sophisticated information. Thus, in some ways, these frameworks can be seen more as complementary than either in agreement or in opposition. Therefore, the lack of connections and consistency in the extant literature between the SOLO taxonomy and the Van Hiele levels regarding student work in any field is, therefore, understandable.

Second, while much supporting research has investigated student geometric understanding through the lens of the Van Hiele levels in various geometric topics (Abdullah & Zakaria, 2013; Clements & Battista, 2003; Clements, Swaminathan, Hannibal, & Sarama, 1999; Hansen, Drews, & Dudgeon, 2014; Spelke, Lee, & Izard, 2010; Van Hiele, 1986, 1999; Vosniadou, 2013; Vosniadou & Skopeliti, 2014; Vosniadou et al., 2015), some studies have questioned if student understanding is adequately assessed in all geometric topics and have argued that, while the Van Hiele levels primarily are suitable for Euclidean geometry, they should not be applied to analytic geometry (Abdullah & Zakaria, 2013). While the SOLO taxonomy (Biggs & Collis, 1982) can be applied to assess student activity in any content area, in mathematics, it has been principally reserved for the consideration of student algebraic understanding. However, this may be primarily due to the Van Hiele levels being so strongly established in the domain of geometry. Thus, very few examples have been found of applying the SOLO taxonomy to student learning in geometry.

Third, since analytic geometry can be seen as existing at the intersection of algebra and geometry, it is highly likely that both the Van Hiele levels and the SOLO taxonomy can be applied as lenses to investigate student understanding. Furthermore, it may be that applying these frameworks jointly might introduce novel dimensions to the analysis of student work, including possibly shedding light on constructs not previously recognized. In order to do so, however, it may be beneficial to connect the Van Hiele levels with an additional framework through which to assess student understanding and the SOLO taxonomy with additional frameworks through which to recognize student activity while learning. To this end, we simultaneously consider the Van Hiele levels (Van Hiele, 1986, 1999), register and domain knowledge (Hiebert, 1988), domain register knowledge (Adu-Gyamfi & Bossé, 2014; Adu-Gyamfi, Bossé, & Chandler, 2017), the SOLO Taxonomy (Biggs, 1999; Biggs & Collis, 1982), syntactic and semantic elaborations (Kaput, 1987a, 1987b, 1989), and isomorphic, transcendent, and mixed connections (Adu-Gyamfi, Bossé, and Lynch-Davis, 2019). All of these frameworks are independently discussed below. After which, we will consider the complementary nature of these frameworks.

Theoretical Frameworks

Mathematical Understanding

Van Hiele levels of geometric understanding. The Van Hiele (1986, 1999) framework is primarily characterized by two main features: a definition of levels of student cognitive understanding of geometry and descriptors of processes through which students progress through the levels (Abdullah & Zakaria, 2013; Clements & Battista, 2003; Clements et al., 1999; Hansen, Drews, & Dudgeon, 2014; Kospentaris , Vosniad, Kazi , & Thanou, 2016; Lim, 2011; Lynn, 2010; Shtulman & Valcarcel 2012; Spelke, Lee, & Izard, 2010; Vosniadou, 2013; Vosniadou & Skopeliti, 2014; Vosniadou et al., 2015). The five Van Hiele levels are defined as:

Level 0 – Visualization. Young students classify geometric shapes by judging their holistic appearance rather than consider properties possessed by the shape.

Level 1 – Analysis. Students recognize some properties of various geometric shapes. While the properties of the shape take on greater importance than the shape itself, the recognized properties are collected but not prioritized. Relationships among the properties are not recognized and concise definitions cannot be formed noting necessary and sufficient conditions. While student can often reason inductively, they cannot yet reason deductively through proofs or ideas.

Level 2 – Abstraction. Geometric properties are the central cognitive construct; they are ordered, deductively connected, and related. Shapes are sorted by typology. Simple arguments and proofs can be followed, and students understand necessary and sufficient conditions leading to writing concise definitions. Lacking an understanding of an axiom system and the meaning and use of deduction, they cannot develop their own deductive proof.

Level 3 – Deduction. Students hold deduction as the central cognitive construct and can independently construct formal geometric proofs at a secondary school level. They understand the axiom system associated with Euclidean geometry but cannot comprehend non-Euclidean geometry.

Level 4 – Rigor. Students possess a mathematician's level of understanding of geometry. The central cognitive construct is that of deductive and flexible axiom systems. Learners can interchangeably study, understand, and apply Euclidean and non-Euclidean geometries.

Five properties are noted in the Van Hiele levels. The levels follow a fixed hierarchical sequence; hold adjacency, where intrinsic (subconscious) properties at one level become extrinsic (conscious) only at the next level; hold distinction, where each level possesses its own relational nomenclature; demonstrates separation, where interlocutors at different levels may use similar language with different meanings; and define phases through which teachers can guide students from one level to another on a given topic. These phases include: *information* or *inquiry*, where teachers present a new idea and students begin to discover structure and work with the new idea; *guided* or *directed orientation*, where students complete teacher-guided tasks to explore implicit relationships and properties associated with new concept; *explication*, where students communicate their ideas and learn contextualized linguistic symbols; *free orientation*, where students engage in increasingly complex tasks designed for them to connect properties, contextualize relationships, and develop fluency in the material; and *integration*, where students synthesize and communicate new ideas.

Register, domain, and domain register knowledge. It is necessary to distinguish between a mathematical representation, the ideas encoded in the representation, and the superset of ideas in which the encoded ideas reside. These supersets are often denoted the reference domain, which is similar to Kaput's (1987a, 1987b, 1989) represented world and Duval's (1999, 2006) idea of the knowledge object. The mathematical domains associated with this study include geometry, algebra, and analytic geometry. Mathematical representations must be distinguished from domains. Graphs, equations, lines, and coordinates are all representations which connect to the domain of analytic geometry. Notably, many opine that a representation is meaningless without being connected to a domain (e.g., Duval, 1999, 2006; Goldin, 1987; Hiebert, 1988; Kaput, 1987a, 1987b, 1989; Steinbring, 2006; von Glasersfeld, 1987).

Consistent with Duval (1999, 2006), Hiebert (1988) denotes a representation system as a register. He then distinguishes register knowledge (RK) (understanding the characters, operators, conventions, and set rules of the representation) and domain knowledge (DK) (understanding the domain(s) to which the registers are connected) and argues that, for students to successfully work with representations, they must possess both knowledge realms. Adu-Gyamfi, Bossé, and Chandler (2015, 2017) additionally recognize domain register knowledge (DRK), the intersection of RK and DK. DRK is the understanding of how the domain provides information regarding the representation and vice versa.

Mathematical Activity

The SOLO taxonomy. The SOLO Taxonomy (Biggs, 1999; Biggs & Collis, 1982) is a hierarchical framework that describes five levels of student understanding as they progress through increasing complexity. These levels include:

Pre-structural. Students attempt tasks but do not do so appropriately because they do not adequately understand the material, use overly simplistic heuristics, and often focus on irrelevant aspects of the problem.

Uni-structural. Student responses focus on only one relevant aspect of the problem.

Multi-structural. Student responses focus on several relevant aspects of the problem, but these aspects are addressed disjointedly and additively.

Relational. Relevant aspects of the problem synthesize into a coherent whole leading to an adequate understanding of the topic.

Extended abstract. The synthesized, conceptual whole may be extended to higher levels of abstraction and generalized to a new topic or area.

The SOLO taxonomy also recognizes the intermediate stages of pre-structural to uni-structural, uni-structural to multi-structural, multi-structural to relational, and relational to extended abstract. Through this structure, learning and student experiences with increasingly complex notions are seen as hierarchical. While the SOLO taxonomy addresses student interactions with progressively complex notions in any realm, it has long been adopted as a lens through which to investigate student algebraic understanding.

Syntactic and semantic elaborations. Numerous studies recognize students' need to decode, encode, and selectively combine characters or signs representations in order to correctly interpret representations (Bossé, Adu-Gyamfi, & Chandler, 2014; Brown, Bossé, & Chandler, 2016; Kaput, 1987a, 1987b, 1989; Lesh, Post, & Behr, 1987). Kaput (1987a, 1987b, 1989) identifies different ways in which students interact with a representation: syntactic elaboration (interacting with a representation by directly manipulating the local symbols in the representations without reference to the meaning of the idea represented) and semantic elaboration (interacting with a representation based on the global ideas represented, rather than the symbols themselves).

Isomorphic, transcendent, and mixed connections. While the importance of making connections in the process of mathematical learning are commonly discussed (e.g., Ainsworth, 1999; Brown, Bossé, & Chandler, 2016; Even, 1998; Janvier, 1987; Kaput, 1989; Steinbring, 1997). The literature recognizes three types of connections, which Adu-Gyamfi, Bossé, and Lynch-Davis (2019) denote as isomorphic, transcendent, and mixed connections. Employing different nomenclature, many recognize that isomorphic connections are made locally and syntactically between characteristics of different representations (see Ainsworth, 1999; Even, 1998; Janvier, 1987; Kaput, 1989; Steinbring, 1997). Transcendent connections are made between a representation and one or more respective domains (see Goldin, 1987; Hiebert, 1988; Kaput, 1987a, 1987b; Thompson, 1994; von Glasersfeld, 1987).

Ainsworth (1999) believes that students must use RK to investigate a representation and, simultaneously, employ DK to contextually interpret representations, and Adu-Gyamfi, Bossé, and Chandlers (2017) and Adu-Gyamfi and Bossé (2014) distinguish DRK as the conceptual glue between the two. Even (1998) opines that mathematical learning includes being flexible using both global and point-wise connections. Thus, altogether, syntactic and semantic elaboration (Ainsworth, 1999; Even, 1998; Kaput, 1987a, 1987b), RK and DK (Hiebert, 1988), and DRK (Adu-Gyamfi & Bossé, 2014; Adu-Gyamfi, Bossé, & Chandler, 2017) suggest that isomorphic and transcendent connections can occur simultaneously. Adu-Gyamfi, Bossé, and Lynch-Davis (2019) denotes these simultaneous connections as mixed connections.

Complementary Frameworks

The five frameworks above were selected because they complement each other in a number of ways. First, the notions of understanding and activity are distinct and consider different elements. Indeed, it is uncertain if heightened understanding leads to more advanced actions or vice versa.

Second, the understanding frameworks (Van Hiele and Hiebert) address understanding in different ways. While the Van Hiele levels focus on geometric understanding, the register framework considers any mathematical concept articulated through any representation and contextualized in any mathematical domain. Thus, while Hiebert's framework can be macroscopically applied to wider fields of mathematics, the Van Hiele levels can provide a microscopic view in the context of geometry.

Third, although the activity frameworks (SOLO, elaborations, and connections) can all be applied beyond the scope of mathematics, within the context of mathematics, these frameworks consider different aspects of what students do, use, and make. For instance, the SOLO taxonomy considers what students do when working through increasingly complex concepts. Elaborations define how students use the information at hand, locally or globally, while connections define relationships that the student makes internally and externally among ideas.

Problem Statement

This study seeks to employ a number of simultaneous lenses to investigate student understanding (Van Hiele levels; register, domain, and domain register knowledge) and actions (SOLO taxonomy; syntactic and semantic elaborations; isomorphic, transcendent, and mixed connections) in respect to analytic geometry. These lenses will be used concurrently to help determine if together their combination leads to newly discovered gaps in assessing student understanding. This investigation is contextualized in two scenarios of student discourse: (a) the investigation of the characteristics of a mid-quadrilateral; and (b) developing a parallelogram through analytic geometry.

Methodology

The participants in this study included seven 11th grade precalculus students in a rural high school in a southeastern state in the United States. All participants were enrolled in the same class with the same teacher. Students 1, 2, 3, and 4 were involved in an activity in which they were to analytically investigate the following prompt: *Given any quadrilateral ABCD, with* $\text{midpoint}\left(\overline{AB}\right)=E$, $\text{midpoint}\left(\overline{BC}\right)=F$, $\text{midpoint}\left(\overline{CD}\right)=G$, *and* $\text{midpoint}\left(\overline{DA}\right)=H$. *Use algebra to determine if points E, F, G, and H would define the vertices of a parallelogram* (Scenario 1). Students 5, 6, and 7 were provided the prompt: *Write four equations that would form the sides of a parallelogram that is neither a rectangle nor a rhombus. None of these equations should be a horizontal line* (Scenario 2).

Each of these two groups were given 60 minutes to complete the respective task. These prompts were selected because: they were well situated in the realm of analytic geometry; students had previously encountered learning materials making these prompts appropriately timed in the curriculum; each prompt represented an open-ended problem with multiple entry points and multiple possible heuristics; and each prompt held potential for rich communication when addressed in a collaborative problem-solving situation. Notably, the limited number of participants naturally delimits this study to one more idiosyncratic in nature regarding the seven participants rather than generalizable to a wide swath of people beyond this study.

Participants agreed to be videoed while solving the research problems. In order to discover student thinking and understanding, videos were transcribed (Bogdan & Biklen, 2003) and discourse analysis (Wertsch, 1990; Wertsch, Hagstrom, & Kikas, 1995) was employed to analyze transcripts. Based on the previous literature background, each of the participating researchers independently employed the following codes to the transcripts:

Knowledge Codes:

Van Hiele Levels: Visualization = **Vis**; Analysis = **An**; Abstraction = **Ab**; Deduction = **Ded**; and Rigor = **Ri**;

Register and Domain Knowledge: Register knowledge = **RK**; Domain knowledge = **DK**; and Domain Register knowledge = **DRK**.

Action Codes:

SOLO: Pre-structural = **Pre**; Uni-structural = **Uni**; Multi-structural = **Multi**; Relational = **Re**; and Extended abstract = **Ex**;

Elaborations: Syntactic elaborations = **Syn**; and Semantic elaborations = **Sem**;

Connections: Isomorphic connections = **IC**; Transcendent connections = **TC**; and Mixed connections = **MC**.

Then, collaboratively, the researchers employed code checking (Miles & Huberman, 1994) to ensure agreement among the researchers regarding the transcripts. This led to consensus of the coding of the transcripts.

To assist in analysis of the coded data, the data was graphed in respect to each of the seven individual participants (later seen in Figures 4-10 in Appendix A). This data was then quantified in the following manner.

IC = 1, TC = 2, and MC = 3;
Syn = 1 and Sem = 2;
Pre = 1, Uni = 2, Multi = 3, Re = 4, and Ex = 5;
RK = 1, DK = 2, and DRK = 3; and
Vis = 1, An = 2, Ab = 3, Ded = 4, and Ri = 5.

This allowed for each code to be considered distinctly and each group of codes to be enumerated in a naturally ascending order – following a pattern similar to the Van Hiele levels and the SOLO framework, although the latter is generally not enumerated. In a parallel and similar manner, although again not generally enumerated, MC, Sem, and DRK are all considered more sophisticated thinking than the other associated knowledge and action dimensions. This enumeration of the coding allowed mathematical methods which could simultaneously aid in cross case analysis and diminish the significance of frequency. Table 1 depicts the formulas that were used to quantify the data and arrive at scores for each framework, for combined action and knowledge frameworks, and for a total score. Notably, calculations are based on averages. In these calculations the following codes received the respective values. Respective scores for each participant are later provided in Table 2.

Arguably, the selected enumeration system and calculations demonstrated in Table 2 warrant a disclaimer of limitations regarding the possible results. Since frameworks are being considered both individually and collectively in this study and frameworks are being enumerated to be assessed quantitatively, a myriad of different calculations could have been employed in this study. Any different calculation schema could produce somewhat different or distorted findings. The calculation schema employed in this study attempted to avoid floor/ceiling effects and not allow any knowledge or action to overpower the other dimensions.

Table 1.

Formulas for Quantified Scores

Student	Framework Score	Action and Knowledge Score	Total Score
Student #	$\text{Con} = \frac{\#\text{IC}+(2\#\text{TC})+(3\#\text{MC})}{\#\text{IC}+\#\text{TC}+\#\text{MC}}$ $\text{El} = \frac{\#\text{Syn}+(2\#\text{Sem})}{\#\text{Syn}+\#\text{SemC}}$ $\text{SO} = \frac{\#\text{Pre}+(2\#\text{Uni})+(3\#\text{Multi})+(4\#\text{Re})+(5\#\text{Ex})}{\#\text{Pre}+\#\text{Uni}+\#\text{Multi}+\#\text{Re}+\#\text{Ex}}$ $\text{Dom} = \frac{\#\text{RK}+(2\#\text{DK})+(3\#\text{DRK})}{\#\text{RK}+\#\text{DK}+\#\text{DRK}}$ $\text{VH} = \frac{\#\text{Vis}+(2\#\text{An})+(3\#\text{Ab})+(4\#\text{Ded})+(5\#\text{Ri})}{\#\text{Vis}+\#\text{An}+\#\text{Ab}+\#\text{Ded}+\#\text{Ri}}$	$\text{Act} = \frac{\text{Con}+\text{El}+\text{SOLO}}{3}$ $\text{Kno} = \frac{\text{Dom}+\text{VH}}{2}$	$\text{Total} = \frac{\text{Act}+\text{Kno}}{2}$

The totality of these methods helped researchers to grasp trends within the understanding and actions of each participant and develop a cross case analysis of global notions across all the participants.

Results

The following transcripts are in their entirety during the time frames recorded. The transcripts include the codes used by the researchers. Notably, the transcripts are significantly longer than generally provided in similar studies. This has the purposes of: providing fuller contexts to the transcripts; justifying the researchers' application of the respective codes; and providing the full transcripts to other researchers who may wish to analyze the transcripts through yet another framework to extend this investigation.

Throughout these transcripts, Student 1 is identified as S1 and so forth. Additionally, to assist the reader in recognizing the codes in the transcripts, knowledge codes are provided in red font and action codes are provided in blue font.

Scenario 1

Given any quadrilateral ABCD, with $\text{midpoint}\left(\overline{AB}\right)=E$, $\text{midpoint}\left(\overline{BC}\right)=F$, $\text{midpoint}\left(\overline{CD}\right)=G$, and $\text{midpoint}\left(\overline{DA}\right)=H$. Use algebra to determine if points E, F, G, and H would define the vertices of a parallelogram.

S1. [Sketches a rectangle with vertices A, B, C, and D with two sides parallel to the *x*-axis.] An, RK; Uni, Syn
S2. No. This is a rectangle. It says a quadrilateral. An/Ab, RK; Uni/Multi, IC
S1. But a rectangle is a quadrilateral. An/Ab; Rel, IC
S2. Yes, but it is a certain kind. I'm guessing that this problem is for ANY quadrilateral. If we make it too much like a rectangle, things might look like they work when they don't. Ab/Ded, RK/DK; Multi/Rel, Sem, TC
S1. How about this? [Sketches a parallelogram with one set of bases horizontal.] An, RK; Uni/Muti, Syn
S3. If we don't want it to look too much like a rectangle, then I think that we probably don't want it to look too much like a parallelogram either. So, we shouldn't make it look like any special kind of quadrilateral. I think that it shouldn't even look like a parallelogram. And sides shouldn't be either flat or up and down. Ab/Ded, RK/DK; Multi/Rel, Syn/Sem. IC
S1. You mean horizontal and vertical. Syn
S4. What about this? [Sketches a non-simple, closed, four-sided figure as in Figure 1.] An/Ab, RK; Uni, Syn

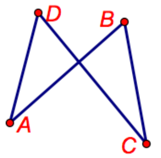

Figure 1

S2. Is that officially a quadrilateral? An, RK; Uni, Syn, IC
S4. It has four sides. An, RK; Uni, RK, Syn
S2. But it crosses. An, RK; Uni, Syn
S1. Does it matter?
S3. I think so. The exact definition of any polygon is a simple, closed, figure. This one is closed, since it ends where it started. But it is not simple. An/Ab, RK/DK; Multi, Syn, IC
S2. What's simple, again?
S3. Simple means that it doesn't cross. Since this crosses, it is not officailly a polygon. So, it is not officially a quadrilateral. Ded, RK/DK; Multi, Syn/Sem, IC
S4. I hope you're right; that this isn't a polygon. I would have trouble the with idea of opposite sides and angles. An/Ab, RK; Uni/Mult, Syn
S2. We could just move the point around and uncross the sides. An; Uni, Syn
S1. What about this? [Sketches a scalene convex quadrilateral.] An/Ab, RK; Uni, Syn
S2. That works. But, even better… [Sketches a scalene concave quadrilateral.] Ab/Ded, DRK; Multi, Syn/Sem, IC
S3. I like that. It certainly isn't any special quadrilateral, but it is a quadrilateral. And it is even concave, which really makes it strange. Ab, RK/DK; Multi, Syn/Sem, IC
S4. Are we sure that it is a quadrilateral if it sinks in? An; Pre/Uni, Syn
S1. It fulfills the definition of a quadrilateral: a four-sided polygon. An/Ab, RK; Uni, Syn, IC

S2. [Labels the vertices, makes tick marks, and labels the midpoints. He produces Figure 2.] An, RK; Uni, Syn

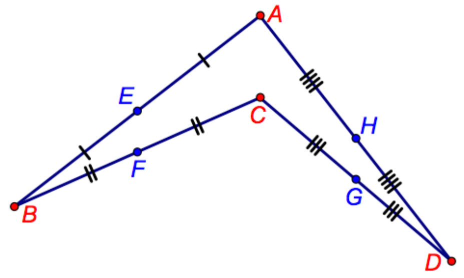

Figure 2

S1. We're trying to determine if *EFGH* makes a parallelogram. Ab/Ded, RK; Multi, Syn

S4. So, what do we do? There are no lines. Are we supposed to connect the dots? [Pointing to the tick marks.] What are these marks? Vis; Pre, Syn

S3. They show those segments are congruent – the same size. So, if *AE* is congruent to *BE*, then *E* is the midpoint of *AB* and *F* is the midpoint of *BC*. I think that we make the segments. [He points from *E* to *F*, from *F* to *G*, from *G* to *H*, and from *H* to *E*. He sketches the segments and constructs quadrilateral *EFGH*.] An/Ab, RK; Multi, Syn

S1. Nothing crosses. It looks like a parallelogram. Vis, RK; Pre/Uni, Syn

S2. It could be close but not exact. We can't just use our eyes. Ab/Ded, RK; Multi, Sem

S3. And the problem said that we need to use algebra.

S4. There aren't any equations. An; Pre/Uni, Syn

S1. But we need to use algebra.

S2. How?

S4. We don't even have *x*- and *y*-axes. An; Pre/Uni, Syn

S1. Wait. We can make a coordinate system. Ab/Ded, DRK; Multi/Re, Sem, TC

S4. What do you mean?

S1. Can't we just give one point a value and go from there? Ab/Ded, DRK; Uni/Multi, Syn/Sem, TC

S2. But how will we know where all the points are? Let's say that $A = (10{,}10)$. What are the coordinates of *B*, *C*, and *D*? Ab, RK; Uni/Multi, Syn, IC

S1. If that's *A*, then we know that *C* is less than *A*. Ab/Ded, RK; Uni, Syn

S2. Like *C* may be (10, 8). An/Ab, RK; Uni, Syn

S4. I'm not sure that *C* is straight below *A*. I think it is a little off. *C* might be closer to (10.2, 8). An, RK; Uni, Syn

S1. Great point. We don't know exactly where *C* is. But we don't know exactly where any point is. Any coordinates would only be a guess. It doesn't seem to make sense that we are supposed to just guess about each point's location? Ab/Ded, DRK; Multi/Re, Sem, TC

S4. Maybe we can just make a grid and see where the points are on the grid. Ab, RK/DK; Uni/Multi, Syn/Sem, TC

S1. That would still be guessing. Sem

S3. Instead of guessing could we use variables? Ab/Ded, RK/DK; Uni/Multi, Syn/Sem, TC

S4. For what?

S3. For the coordinates. An/Ab, DRK

S4. You mean like *A* is *x* and *B* is *y*? Ab, RK; Uni, Syn, IC

S3. Kinda. But *A* and *B* could already be considered variables. I mean that we need variables for the coordinates. Ab/Ded, RK; Uni/Multi, Syn

S4. What do you mean?

S1. Like $A = (x, y)$ and $B = (x, y)$. Ab/Ded, DRK; Multi, Syn

S3. But then all the coordinates would have the same variables. So, we probably want something like $A = \left(x_a,\ y_a\right)$ and $B = \left(x_b,\ y_b\right)$. Ab/Ded, DRK; Multi/Re, Syn/Sem, IC

S4. What does that mean?

S2. It's like $A = (x, y)$, but now we can tell that x_a is the *x*-value of *A* and y_c is the *y*-value of *C*. Ab/Ded, DRK; Multi/Re, Syn/Sem, IC

S4. Ok. But the *y*-value of *C* is bigger than the *y*-value of *A*. An/Ab, RK; Uni, Syn

S3. For our picture, *A* is above *C*. But for other cases, *C* might be above *A*. So, I don't think it really matters. Ab/Ded, RK/DK; Multi/Re, Syn/Sem, IC

S4. But it does on this picture. RK; Syn

S3. This is the picture we chose. We could have chosen any other. We just need for *A* and *C* to be nonadjacent vertices. Ab/Ded, RK/DK; Multi/Re, Sem

S2. Nonadjacent?

S3. Opposite vertices: they don't share a side.

S4. So, we make a picture, but we're not supposed to use it? An; Pre/Uni, Syn

S1. We're using it. But we know that the vertices are just variables. As long as it forms a quadrilateral, we're ok. Ab/Ded, RK/DK; Multi/Re, Sem, MC

S3. So, let's go with something like $A=(x_a,\ y_a)$, $B=(x_b,\ y_b)$, $C=(x_c,\ y_c)$ and $D=(x_d,\ y_d)$. Ab/Ded, DRK; Multi/Re, Syn, IC

S4. Now the vertices aren't exact points. They can be anywhere? Syn

S2. Yes, as long as it makes a quadrilateral. That is its power. It can be *any* quadrilateral. That's what we want. Ded, RK/DK; Multi/Re, Sem, TC

S4. But, if the points are anywhere, because they are variables, couldn't they make a crisscross quadrilateral? Ab, RK; Uni/Multi/Re, Syn, IC

S3. I guess that we could get that, but let's just work through this and see what happens. An/Ab/Ded; Multi, Syn

S4. But we still don't have any algebra.

S1. We do have algebra. We have variable coordinates. That's algebra. Ab, RK/DK; Multi, Sem MC

S4. We don't have equations. An; Pre/Uni, Syn

S1. But we might, if we go further. Sem, TC

S2. We need to find the coordinates of *E*, *F*, *G*, and *H*. Those are midpoints. An/Ab, RK; Multi, Syn, IC

S3. What about $E=(x_e,y_e)$, $F=(x_f,y_f)$, $G=(x_g,y_g)$, and $H=(x_h,y_h)$? Ab/Ded, DRK; Multi/Re, IC

S4. That's getting to be a lot of letters. Syn

S1. If, we use that, then *E*, *F*, *G*, and *H* are all *any* points. We don't know if they are midpoints. An, RK; Pre/Uni, Syn

S2. Aren't they midpoints if we say they are midpoints? An; Uni, Syn

S3. Well, not quite. So, I'm thinking…

S4. Can we make sure they are midpoints?

S3. I think that, instead of making more variables, we need to connect the midpoints to the original points. Ab/Ded; Multi/Re, Sem, MC

S1. How?

S2. If they are midpoints, then can't we use the midpoint formula? Ab/Ded, RK/DK; Multi/Re, Sem, TC

S4. I forget that one.

S2. With two points, (a,b) and (c,d), the midpoint should be $\left(\frac{a+c}{2},\frac{b+d}{2}\right)$. An/Ab, RK; Uni, Syn

S3. Yes. The midpoints are the averages of the *x*- and *y*-values of the coordinates. Ab/Ded, RK; Multi/Re, Sem, IC

S4. But *A*, *B*, *C*, and *D* are points. How can you get averages of points? An/Ab; Uni, Syn

S2. What do you mean? Those are coordinates. An, RK; Pre/Uni

S4. No. We said that they are points.

S1. I'm confused. They are coordinates. Vis/An; Uni

S3. Wait. I got it. [Speaking to Student 4] Capital *A*, *B*, *C*, and *D* are points, but small *a*, *b*, *c*, and *d* are coordinates. That's what you are getting wrong. Ab, RK; Multi/Re, Syn/Sem

S2. Oh, that's what was going on. I didn't get it.

S4. Coordinates to what? An; Pre/Uni

S3. Just random coordinates. They are just examples. They have nothing to do with points *A*, *B*, *C*, and *D*. Ab/Ded, DRK; Multi/Re, Sem, TC

S1. [Speaking to Student 4] We probably should have used other letters to keep from being confused. Ab/Ded, DRK; Multi/Re, Syn/Sem

S2. So, rather than *E* being (x_e,y_e), we should probably make $E=\left(\frac{x_a+x_b}{2},\frac{y_a+y_b}{2}\right)$. That would guarantee that *E* is the midpoint between *A* and *B*. Ab/Ded, DRK; Multi/Re, Syn/Sem, MC

S4. That makes more sense. But we still have a lot of letters. An; Uni

S2. We actually have more: $F=\left(\frac{x_b+x_c}{2},\frac{y_b+y_c}{2}\right)$, $G=\left(\frac{x_c+x_d}{2},\frac{y_c+y_d}{2}\right)$, and $H=\left(\frac{x_d+x_a}{2},\frac{y_d+y_a}{2}\right)$. Ab/Ded, DRK; Multi/Re, Syn, IC

S3. It's more letters, but now every variable is connected to our original points and coordinates. Ab/Ded, DRK; Multi/Re, Syn/Sem, IC

S1. We must be on the right path. This is starting to look like algebra. Vis/An; Pre/Uni, Syn

S3. Well at least we now have variable coordinates for the midpoints.

S4. Now what do we do with them?

S1. I think that we are kinda back to the beginning. We need to know what makes a parallelogram. An/Ab/Ded; IC

S2. Opposite sides are parallel. An/Ab, RK; Uni, Syn
S3. Opposite sides are equal. An/Ab, RK; Uni, Syn
S1. A quadrilateral with two pairs of opposite parallel sides is the definition. So, I am ok with opposite sides are parallel. Opposite sides are equal is tougher. A rhombus and a square have equal opposite sides. They are parallelograms. But, they are kinds of parallelograms. I think that we want any parallelogram. We need a better definition of a parallelogram using equal sides. Ab/Ded, DK; Multi/Re, Sem, MC
S3. My bad. I mean two pairs of opposite congruent sides. An/Ab/Ded, DK; Multi/Re, Sem
S4. Opposite angles are equal. Ab; Uni, Syn, IC
S2. That is right. But we don't know anything about the angles in our quadrilateral. I don't think we should use angle stuff. Ab/Ded, RK; Uni, Syn
S1. So, we either go for length or congruency? Which is easier? Ab/Ded, DK; Multi/Re, Syn, TC
S2. Well, length would take some work. We would need to do something like: the distance from *E* to *F* would be $\sqrt{\left(\frac{x_a+x_b}{2}-\frac{x_b+x_c}{2}\right)^2+\left(\frac{y_a+y_b}{2}-\frac{y_b+y_c}{2}\right)^2}$. Ab/Ded, DRK; Multi/Re, Syn/Sem, MC
S4. That's pretty ugly.
S3. But it can be simplified: $\sqrt{\left(\frac{x_a+x_b-(x_b+x_c)}{2}\right)^2+\left(\frac{y_a+y_b-(y_b+y_c)}{2}\right)^2}=\sqrt{\left(\frac{x_a-x_c}{2}\right)^2+\left(\frac{y_a-y_c}{2}\right)^2}=\sqrt{\frac{(x_a-x_c)^2}{4}+\frac{(y_a-y_c)^2}{4}}=\sqrt{\frac{(x_a-x_c)^2+(y_a-y_c)^2}{4}}=\frac{\sqrt{(x_a-x_c)^2+(y_a-y_c)^2}}{2}$. Ded, DRK; Multi/Re, Syn, IC
S4. Wow.
S3. I love doing this.
S1. It all looks right.
S2. Should we expand what's under the radical? An; Uni, Syn
S3. Since all the *x*'s and *y*'s are just numbers, I don't think that we should expand it. It will be simple arithmetic. Let's do the same thing for another side. An/Ab/Ded, RK; Uni, IC
S2. Let's find the length from *F* to *G*. An/Ab, RK; Uni, IC
S1. Why not find the length of *GH*. At least that side would be opposite *EF* that we just found the length of. Ab/Ded, DK; Multi/Re, Sem, MC
S3. Let me do it. [After doing some rewriting of expressions he produces] $\frac{\sqrt{(x_c-x_a)^2+(y_c-y_a)^2}}{2}$. An/Ab, RK; Uni/Multi, Syn, IC
S2. So, those two opposite sides are equal. Ab/Ded, RK; Multi/Re, IC
S1. Or congruent.
S4. We have opposite sides equal. So, do we have a parallelogram? Ab/Ded, RK; Uni, Syn
S1. Not yet. We have one pair of opposite sides congruent; but we need both pairs of opposite sides to be congruent. Ab/Ded, DK; Multi/Re, Sem, IC
S4. But doesn't parallelogram mean parallel? An, RK; Uni, Syn
S2. Yes. But we have options: if either both pairs of opposite sides are parallel or both pairs of opposite sides are equal, then it is a parallelogram. Ded, DK; Multi/Re, Sem, IC
S3. But there is another option: one pair of opposite sides are both parallel and congruent. Since we already have the lengths of *EF* and *GH* and calculating the slope of those two lines may be easier than calculating lengths again, we should just check their slopes. Ab/Ded, DRK; Multi/Re, Syn/Sem, MC
S4. Can we do slopes? We don't have lines. An/Ab; Pre, Syn
S1. But we have line segments. These can be extended into lines. Ab/Ded, RK/DK; Uni, Sem, TC
S4. We wouldn't know where they cross the *y*-axis. We don't even have axes. An, RK; Uni, Syn
S2. We don't need the equation of the line. We only need the slopes. Ab/Ded, RK/DK; Multi/Re, Sem, TC
S4. How can we have slopes without axes? An; Pre/Uni, Syn
S3. We can get the slopes from the points. For *EF* we have the slope $\frac{\frac{y_a+y_b}{2}-\frac{y_b+y_c}{2}}{\frac{x_a+x_b}{2}-\frac{x_b+x_c}{2}}=\frac{y_a+y_b-(y_b+y_c)}{x_a+x_b-(x_b+x_c)}=\frac{y_a-y_c}{x_a-x_c}$. And the slope of *GH* [after similar calculations] is $\frac{y_c-y_a}{x_c-x_a}$. Ded, DRK; Multi/Re, Syn/Sem, MC
S1. Those are different. Syn
S2. I don't think so. I think we can do: $\frac{y_a-y_c}{x_a-x_c}=\frac{-1}{-1}\left(\frac{y_a-y_c}{x_a-x_c}\right)=\frac{-y_a+y_c}{-x_a+x_c}=\frac{y_c-y_a}{x_c-x_a}$. So, they are the same. Ab/Ded, DRK; Multi/Re, Syn/Sem
S3. Now we have a pair of opposite sides where the sides are both parallel and congruent. That gives us a parallelogram. Ded, DK; Multi/Re, IC
S4. We used algebra. But I got lost.

S3. Hey, wait a minute. I think that we found more than we needed to. I think that [Student 4] had an important idea. We were concerned that the quadrilateral was a true quadrilateral and not a crisscross quadrilateral. But if we use variable points, we can't guarantee that we don't have a crisscross quadrilateral. So, I think that the midpoints of either a real quadrilateral or a crisscross quadrilateral makes a parallelogram. Ab/Ded, DRK; Multi/Re, Sem, MC

Scenario 2

Write four equations that would form the sides of a parallelogram that is neither a rectangle nor a rhombus. None of these equations should be a horizontal line.

S5. How are we supposed to do that?
S6. I think that we simply need to start with a line. Any line that is not horizontal or vertical. How about $y = 3x + 2$? Ab/Ded, RK/DK/DRK; Multi, Syn/Sem, TC
S7. That works. But might other equations work also? Ab/Ded, RK/DK; Multi, Sem
S5. Of course. That's just one. We could also use $y = -2x + 5$. But that's just one other example. An/Ab, RK; Uni, Syn
S7. How about $y = mx + b$? That would get us all the possible equations. Ab/Ded, RK/DK/DRK; Multi/Re, Syn/Sem, IC
S6. But that might make a horizontal or vertical line. Ab/Ded, RK/DK; Multi/Re, Syn/Sem, TC
S5. We can't have horizontal or vertical lines. Ab/Ded, DK; Multi, Sem
S7. We need conditions on m and b. How about: $0 < m < \infty$? Ab/Ded, DRK; Multi/Re, Syn/Sem, MC
S5. Why?
S6. If $m = 0$, then we have a horizontal line and if $m = \infty$, then we have a vertical line. Ab/Ded, RK/DK; Multi/Re, Syn/Sem
S5. How can we have $m = \infty$? An; Uni, Syn
S6. If we have something over zero, we have infinity. Ab, RK/DK; Syn
S5 How do we get something over infinity? An; Pre
S7. When we calculate the slope, we have change of y over change of x. If we have two points with the same x-values, like $(3,5)$ and $(3,-2)$, we get a slope of $\frac{5-(-2)}{3-3} = \frac{7}{0} = \infty$. Ab/Ded, DRK; Multi/Re, Syn/Sem
S5. Ok. And what about b? An, RK; Uni, Syn
S7. I think that b can be any real number. It's just a y-intercept. So, I'm ok with $y = mx + b$ with that condition on m. Ab/Ded, RK/DK/DRK; Multi/Re, Sem, MC
S5. I don't get it. Why can b be any value? An, RK; Uni, Syn
S6. [Gesticulating with his arms and hands.] If the line has some slope, then we can just shift if up or down. This keeps the slope the same, but changes the y-intercept. But any of those y-intercepts could work. Ab/Ded, RK/DK; Multi/Re, Syn/Sem, IC
S5. But what about negatives? Couldn't the line be negative? An; Pre/Uni, Syn
S6. You mean that the slope is negative. It can. But I don't think that it has to be. It we keep it positive, then the connecting side may have a negative slope. An/Ab, RK/DK; Multi, Syn/Sem, IC
S7. I'm ok with all of this. What about another side? An/Ab, RK; Uni, IC
S5. Which other side should we start with? An, RK; Uni, Syn
S6. I think the opposite side would be easiest. The two sides are parallel. So, they have the same slope. Ab/Ded, RK/DK; Multi/Re, Sem, IC
S7. But they need to have different y-intercepts so that they are not both the same line. What about $y = mx + c$? Ab/Ded, RK/DK/DRK; Multi/Re, Syn/Sem, MC
S5. Do we need to say that the two m's are the same? Ab, RK; Uni, Syn
S6. No. If we use the letter again, it is the same. But we need some conditions for c. Ab/Ded, RK/DK; Multi/Re, Syn
S7. Can we say that c is any real number like we did for b? Ab/Ded, DRK; Multi/Re, Syn/Sem, IC
S5. I think so.
S6. No. c cannot be the same as b. Or the two lines would be the same. I think we need the condition $c \neq b$. Ab/Ded, DRK; Multi, Syn/Sem, IC
S7. Right.
S5. We now have two lines: $y = mx + b$ and $y = mx + c$. And I think that they are opposite lines. And they are lines, not segments. Shouldn't we shorten them some? An/Ab, RK/DK; Uni/Multi, Syn, IC

S7. We cannot shorten them until we know what the other sides are. Where they intersect the other two sides will make them shorter. Ab, RK; Multi/Re, Syn/Sem, IC
S6. So, what about one of the other lines. Since it is a line, we should be able to use the $y = mx + b$ idea again. An/Ab, RK; Uni, Syn
S7. But we can't use m. We need another variable. Ab/Ded, RK; Multi, Syn
S5. Does it matter what we use? How about an n? RK; Uni, Syn
S6. So, what about $y = an + b$? Ab, RK; Multi, Syn
S7. I think the n is good as long as we make sure that $n \neq m$. Ab/Ded, RK; Uni/Multi, Syn, IC
S5. Is the b ok? An, RK; Uni, Syn
S6. Can b be the same as the old b or c? An/Ab, RK; Uni, Syn, IC
S7. I think so. I don't think that it matters what the y-intercept is on this line. Ab/Ded, RK; Multi/Re, Syn
S5. Why did we need a different letter from m? An/Ab, RK; Uni, Syn
S6. We used m for the first two lines because they are parallel and needed to have the same slope. But this line needs to be a different slope. Ab/Ded, RK/DK; Multi/Re, Syn/Sem, IC
S5. Why?
S6. So that the lines will intersect. Ab/Ded, DK; Multi/Re, Sem,
S7. If lines have different slope, then they are not parallel and they will intersect somewhere. Ab/Ded, RK; Multi/Re, Syn
S5. I got it now.
S6. But maybe we need to call it d. An/Ab/Ded, RK; Uni, Syn
S5. Call what d? An; Pre, Syn
S6. The y-intercept for the third line. Ab, RK; Uni
S7. And since d can be either b or c and any other number, maybe its ok just to say that $-\infty < d < \infty$. Ab/Ded, DRK; Multi/Re, Syn/Sem, MC
S5. We're getting lots of numbers here. An; Pre/Uni
S6. We might have more before we are done. An, RK; Uni
S7. Now we have two parallel lines for two sides and another line that intersects the two parallel lines. Ab/Ded, RK/DK; Multi/Re, Sem, IC
S5. How do we know that the line intersects the first two lines? An
S6. If two lines are parallel, they don't intersect. But if one line intersects one of the two parallel lines, it has to intersect the other. Ab/Ded, DK; Multi/Re, Sem, IC
S5. That's in geometry. Is that the same in algebra? An; Pre/Uni
S7. Yes. So, what about the fourth line? An; Uni
S6. Wait. Did we make a mistake? We said that $0 < m < \infty$. Why can't m be negative? Ab/Ded, DRK; Uni/Multi, Syn
S5. Pointing down? An; Pre/Uni, Syn
S7. Yes, pointing down. I think that the slope can be negative as long as it is not zero and not ∞. Ab/Ded, DRK; Uni, Syn
S6. You mean $-\infty < m < \infty$, with $m \neq 0$. Ab/Ded, DRK; Uni/Multi, Syn
S7. Yup. That works.
S5. But what happens to n? An, RK; Pre/Uni, Syn
S7. It's the same thing as m. Ab; Uni/Multi, Syn/Sem, IC
S5. I thought m and n had to be different. An, RK; Uni/Multi, Syn
S7. They do, but both $-\infty < m < \infty$ and $-\infty < n < \infty$, with both $m \neq 0$ and $n \neq 0$, and $m \neq n$. Ab/Ded, DRK; Multi/Re, Syn/Sem, IC
S6. Looks good. So back to the last line. What about $y = nx + e$? Ab, RK; Multi/Re, Syn/Sem, MC
S7. Since we need the same slope as the third line, the n works. And I'm guessing that you want the e to be different from the d. An/Ab, RK/DK; Multi/Re, Syn/Sem, IC
S6. Yes. Ab/Ded, RK; Multi/Re, Syn
S5. And you want e and d to be different so that we have different lines. Right? An/Ab, RK; Multi, Syn, IC
S6. Right.
S5. We know that the m and n lines intersect. But do we know where? Ab, RK; Uni/Multi, Syn, IC
S7. I was thinking about that. I think that we can set up an equation. I think that, since $mx + b$ is the value of y and $nx + d$ is a y value, we are looking for when these y's are the same. So,.. Ab/Ded, DRK; Multi/Re, Syn/Sem, MC
S6. Yeah, so, $mx + b = nx + d$. That means that $mx = nx + d - b \Rightarrow mx - nx = d - b \Rightarrow (m - n)x = d - b \Rightarrow x = \frac{d-b}{m-n}$. Ab/Ded, DRK; Multi/Re, Syn/Sem, MC

S5. Whoa. What's going on?

S7. We found the x-value where the first and third line intersect. To be more precise, the lines intersect at $\left(\frac{d-b}{m-n}, f\left(\frac{d-b}{m-n}\right)\right)$. Ab/Ded, DRK; Multi/Re, Syn/Sem, MC

S5. Where does the f come from? An; Pre/Uni

S7. Instead of y, I used f. So, I could have written it $\left(\frac{d-b}{m-n}, m\left(\frac{d-b}{m-n}\right)+b\right)$. Ab/Ded, DRK; Multi/Re, Syn, IC

S6. We weren't asked where the lines intersect. We were only asked for the lines to form a parallelogram. I don't want to do more than we need to do. An

S5. Did we do it?

S6. Yes. Our lines are: $y = mx + b$, $y = mx + c$, $y = nx + d$, and $y = nx + e$ with the conditions we had before. Ab/Ded, RK; Multi/Re, Syn, IC

S5. Can we make a picture to be sure? An; Uni

S7. We really don't need one. We know that this would work. Ab/Ded; Multi/Re

S5. But can we make a picture anyway? An, RK; Uni, Syn

S6. That's not easy to do. We used variables for our slopes and intercepts. That means we actually made every case. If we made a picture, we would be looking at just one case. Ab/Ded, RK/DK; Multi/Re, Syn/Sem

S5. But can't we do that anyway? An; Uni

S6. We can. We can even use kinda simple examples. Keeping m's and n's consistent, we can make the lines: $y = 2x - 2$, $y = 2x + 8$, $y = -x + 1$, and $y = -x + 5$. [Figure 3 is produced in a graphing utility.] Ab, DRK; Uni/Multi, Syn/Sem, IC

S5. That looks like a parallelogram to me. But are we sure that a graphing parallelogram is the same as a geometry parallelogram? An, RK; Uni, Syn

S7. I think that they are the same thing, other than when we use the lines we really mean the segments between the intersections of the lines. Ab/Ded, RK/DK; Multi/Re, Syn/Sem, IC

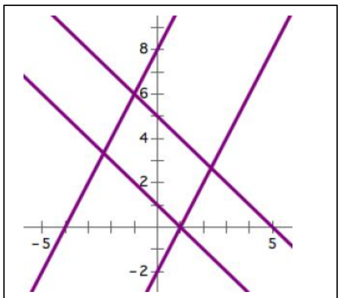

Figure 3

Analysis of Results

The graphs provided in the Appendix depict the increments of observation of coded activities from the transcripts for each participant. The graphs are divided into the two knowledge and three action frameworks previously discussed.

These graphs are sufficient to demonstrate that these seven participants exhibited different levels of knowledge and activity as well as varying frequency of increments of observation over the research activities. Informally, Students 1 and 7 may seem to have higher numbers of increments of observations in codes in the upper ranges of each of the frameworks, possibly indicating more sophisticated understanding and activity associated with the research task. Conversely, Students 4 and 5 seem to have higher numbers of increments in codes in the lower ranges of the frameworks, possibly indicating less sophisticated understanding and activity. The following look more deeply into some of the students' transcripts and codes.

1. In some instances, multiple codes are ascribed even within one framework for a specific observed behavior (e.g., Student 1, increment 24, (An, AB, and Ded) in the Van Hiele Levels). Where such multiple codes exist, this should not be interpreted as disagreement among the researchers doing the coding. Rather all the researchers recognized that particular actions exhibited multiple contextualized meanings and purposes in the mind of the participant.

2. No student remained exactly consistent throughout either knowledge or action frameworks; all students showed variability through the frameworks. Even students who were more consistently higher in the frameworks occasionally had codes in the lower ranges.

In both knowledge and action realms, some students exhibited activity that could be coded at multiple levels in various frameworks.

3. While students with codes notably in the lower levels of the knowledge frameworks (Students 4 and 5) had mostly correspondingly low codes in the action frameworks, students demonstrating codes in the higher levels of in the knowledge frameworks (Students 1, 3 and 7) demonstrated great variability in the levels of action codes.

4. While these transcripts may seem rather lengthy, particularly in respect to more typical reports of qualitative research, they are but snippets in time within a semester-long precalculus class. Nevertheless, these transcripts demonstrate some degree of consistency across each participant and no student is seen to demonstrate upward trends through the frameworks as they progress through the respective research task. However, it is simply wrongly to interpret this lack of upward trend as the student not learning throughout the problem-solving activity. Assessing learning is not the focus or purpose of this study. Rather, it can be stated that, for these study participants, the results of the associated research activity demonstrate little upward movement through the investigated frameworks.

5. While students demonstrating lower levels on the knowledge and activity frameworks (Students 4 and 5) also had significantly fewer increments of observations, the researchers opted to dismiss consideration of frequency overall, in that it could be associated more to a student's shyness in collaborating with others than in their actual knowledge or ability. Employing the calculation methodology defined in Table 1, Table 2 calculates scores for each framework, for combined action and knowledge frameworks, and for a total score and minimizes the effect of frequency.

Table 2.

Quantified Scores Per Participant

Student	Framework Score	Action and Knowledge Score	Total Score	Student	Framework Score	Action and Knowledge Score	Total Score
Student 1	**Con = 1.94** **El = 1.50** SO = 2.28 Dom = 1.81 VH = 3.04	1.91 2.43	2.17	Student 2	Con = 1.53 El = 1.39 SO = 2.88 Dom = 1.53 VH = 3.02	1.93 2.28	2.02
Student 3	Con = 1.48 El = 1.47 **SO = 3.30** Dom = 1.88 VH = 3.26	2.08 2.57	2.33	Student 4	Con = 1.25 El = 1.08 *SO = 1.91* *Dom = 1.08* VH = 2.38	1.41 1.73	1.57
Student 5	*Con = 1.00* *El = 1.05* *SO = 1.91* Dom = 1.12 *VH = 2.26*	*1.32* *1.69*	*1.51*	Student 6	Con = 1.46 El = 1.39 SO = 3.07 Dom = 1.65 VH = 3.23	1.97 2.44	2.21
Student 7	Con = 1.71 El = 1.46 SO = 3.29 **Dom = 1.97** **VH = 3.36**	**2.15** **2.67**	**2.41**				

Discussion

Throughout the remaining portions of this paper, the authors make careful use of three words: knowledge and actions (as related to the knowledge and actions frameworks associated with this study) and understanding (a more global, informal term often associated with a teacher's either intuitive or formal assessment of a student's general comprehension).

This study considers seven student participants' knowledge and actions in the context of analytic geometry through five simultaneous lenses: Van Hiele levels (Van Hiele, 1986, 1999), the SOLO taxonomy (Biggs, 1999; Biggs & Collis, 1982), elaborations (Kaput, 1987a, 1987b, 1989), register and domain knowledge (Adu-Gyamfi & Bossé, 2014; Adu-Gyamfi, Bossé, & Chandler, 2017; Hiebert, 1988), and connections (Adu-Gyamfi, Bossé, & Lynch-Davis, 2019).

The results demonstrate that particular behaviors can be coded both through multiple frameworks and at multiple levels within particular frameworks. This is somewhat in discordance with previous research which most typically equate particular actions to one framework and one level at a time. This study may imply that learners' knowledge, actions, and understanding may be more complex than any single framework may be able to adequately capture. This study may also demonstrate that student work can be simultaneously considered through frameworks assessing mathematical knowledge and frameworks investigating mathematical activity.

In respect to both the knowledge Van Hiele levels (Van Hiele, 1986, 1999) and the action SOLO taxonomy (Biggs, 1999; Biggs & Collis, 1982), higher levels (rigor and extended abstract, respectively) were not recognized within student work. Simultaneously, few student behaviors were coded in the lowest levels (visualization and pre-structural) in these frameworks. Both of these observations may be due to the students being in high school (thus, attaining sufficient level of mastery to be beyond the lowest levels) and not yet reaching their mathematical learning potential (and not yet being at the highest levels). These results may also be attributed to the curriculum, types of activities, and pedagogical practices that the students previously experienced in the classroom.

Although somewhat repetitious, the students with the higher total scores demonstrated some behaviors in some higher levels of these frameworks: abstraction and deduction (Van Hiele, 1986, 1999), domain register knowledge (Adu-Gyamfi & Bossé, 2014; Adu-Gyamfi, Bossé, & Chandler, 2017), multi-structural and relational (Biggs, 1999; Biggs & Collis, 1982), semantic elaborations (Kaput, 1987a, 1987b, 1989), and mixed connections (Adu-Gyamfi, Bossé, & Lynch-Davis, 2019). Students with lower scores tended toward analysis (Van Hiele, 1986, 1999), register knowledge (Hiebert, 1988), uni-structural (Biggs, 1999; Biggs & Collis, 1982), and syntactic elaborations (Kaput, 1987a, 1987b, 1989), and isomorphic connections (Adu-Gyamfi, Bossé, & Lynch-Davis, 2019). Students 1 and 2 showed a few unexpectedly low scores (visualization and pre-structural) in respect to the rest of their work. These results may be attributed to possibly differentiated learning experiences accompanied with self-fulfilling prophecy. It may be that, when teachers recognize students with more sophisticated mathematical understanding, they provide those students with more challenging experiences and that students who are recognized as possessing less sophisticated understanding are provided less challenging experiences, thus polarizing the students through both experiences and effects. More future research is needed regarding these findings, particularly in respect to other mathematical domains.

As previously mentioned, there was no recognized observation of student growth through any of these frameworks over the very limited duration on the respective research tasks. The

Van Hiele levels define five phases through which student progress from one level to another: information or inquiry, directed orientation, explication, free orientation, and integration (Van Hiele, 1986, 1999). Unfortunately, the brief durations of these research tasks provide no evidence of phases which could move students upward in any of the five frameworks investigated, or if the Van Hiele phases would accomplish growth in respect to analytic geometry. Thus, it would be valuable for future research to consider more longitudinal aspects of student growth, including paying attention to the effects of various forms of learning tasks.

Since all previous enumeration and computations methods in this study employed little more than weighted means, the researchers felt it inappropriate to apply more advanced statistical methods to determine correlation or causality or by which to make predictions. Nevertheless, in accordance with previous observations – albeit through informal and nonstatistical conventions – it seems as though for only the students with the lowest and highest total scores, based on the coding of their work, there may be a predictive association between knowledge and action levels. For other students in this study, a predictive association could not be inferred.

Implications

The implications of this study are numerous, although they must be stated cautiously. First, due in part to the complexity of sufficiently assessing student understanding, this and future studies may recognize the need of combining frameworks to investigate student understanding. In time, this may reveal interconnections, redundancies, and discontinuities as well as predictive associations among frameworks. Altogether, these might lead to new tools and lenses through which to investigate student understanding.

Second, the lack of predictive association between knowledge and action frameworks, for what may seem to be the majority of students, may lead researchers to wonder to what degree assessing student understanding through either type of framework paints an adequate picture of the whole of student understanding. Particularly, it is possible that (a) a student with seemingly higher understanding may perform mathematical actions at lower levels than educators may expect and, conversely, (b) observing a student's mathematical actions – even when positive and at higher levels – may not provide an adequate picture of the depth of a student's understanding.

While this second point has implications for the researcher, so, too, it applies to the educator. This point may mean that assessment of student understanding is not a simple task. Do we believe that higher levels of student knowledge lead to higher levels of student activity or vice versa? If not, does that complicate the notion of assessment? And what might this mean to standardized testing? And, how do knowledge, actions and understanding all interrelate? It cannot yet be adequately conceived what these implications may mean to teaching practices and student opportunities to learn. In the future, findings from this study may prove to be all the more impactful on teaching and learning and vice versa. As what students do may not equate to what they know, and vice versa, this could potentially have implications for curriculum development. Furthermore, in respect to this study, we recognize that investigating a small set of students performing one research task is not sufficient for overarching recommendations.

Third, the fact that some students exhibited multiple codes within one framework for a specific observed behavior and codes among multiple frameworks implies that students can be recognized through many dimensions and at numerous positions in those frameworks.

This, again, makes assessment far more complex than it may often be realized. Also, possibly indicated through this study is that students' verbal and representational communications may not adequately reveal their mathematical understanding. This may affect formative assessment and curriculum differentiation for individual students.

Fourth, while no growth was necessarily expected in students' knowledge and actions through singleton research tasks, there may still be the need to investigate phases – such as in the Van Hiele levels – through which students could grow to higher levels in the investigated frameworks. This could also apply to students growing in respect to understanding the content of analytic geometry.

Conclusion

This study investigated student mathematical knowledge and activities in the context of analytic geometry through five simultaneous frameworks. Through this it was determined that much can be learned regarding student work and understanding when considered through multiple lenses.

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Appendix

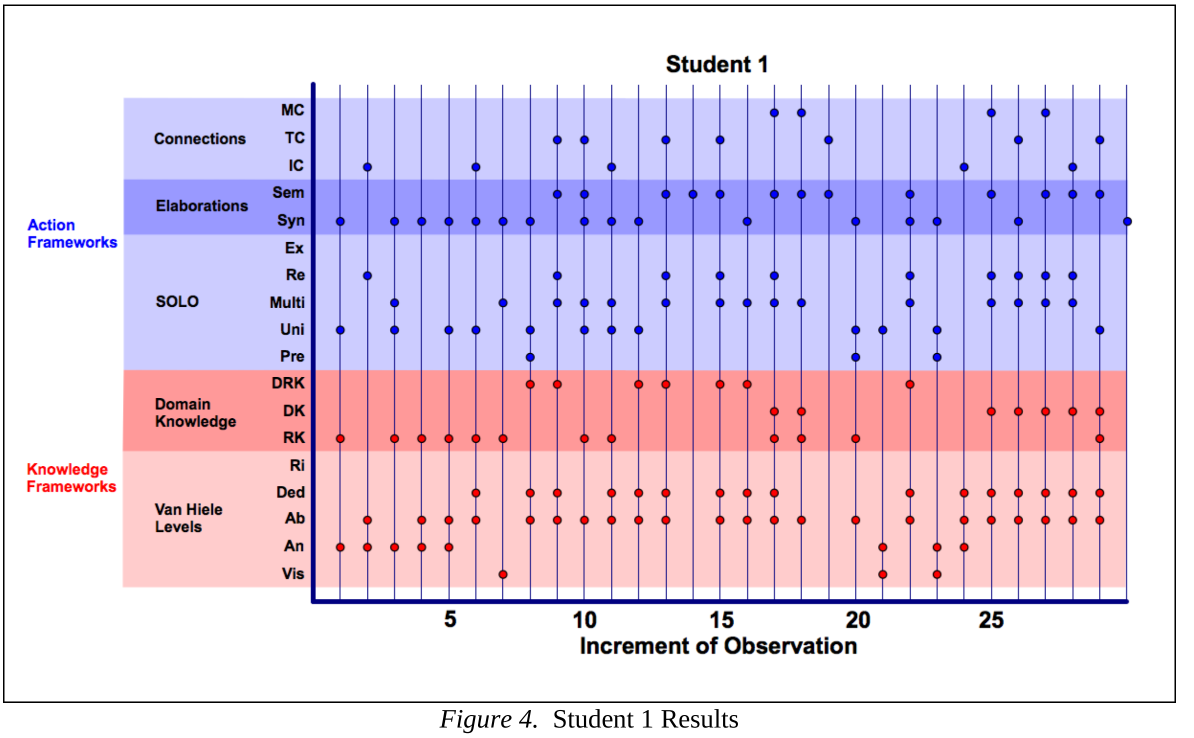


Figure 4. Student 1 Results

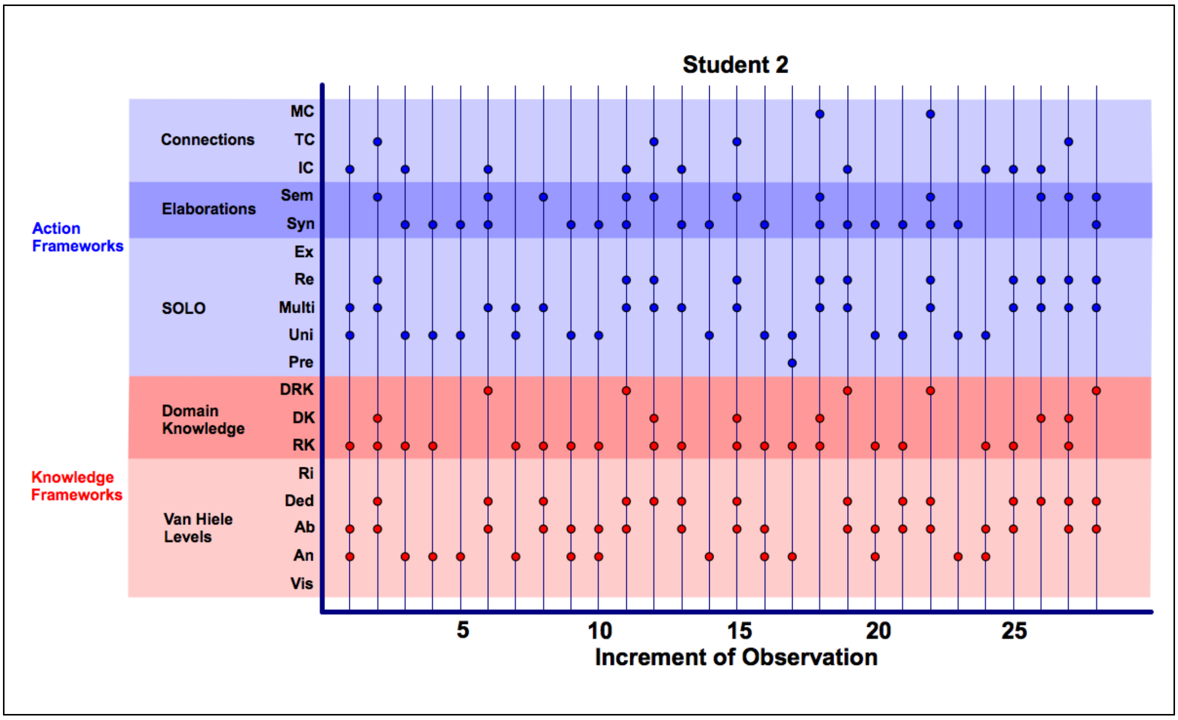


Figure 5. Student 2 Results

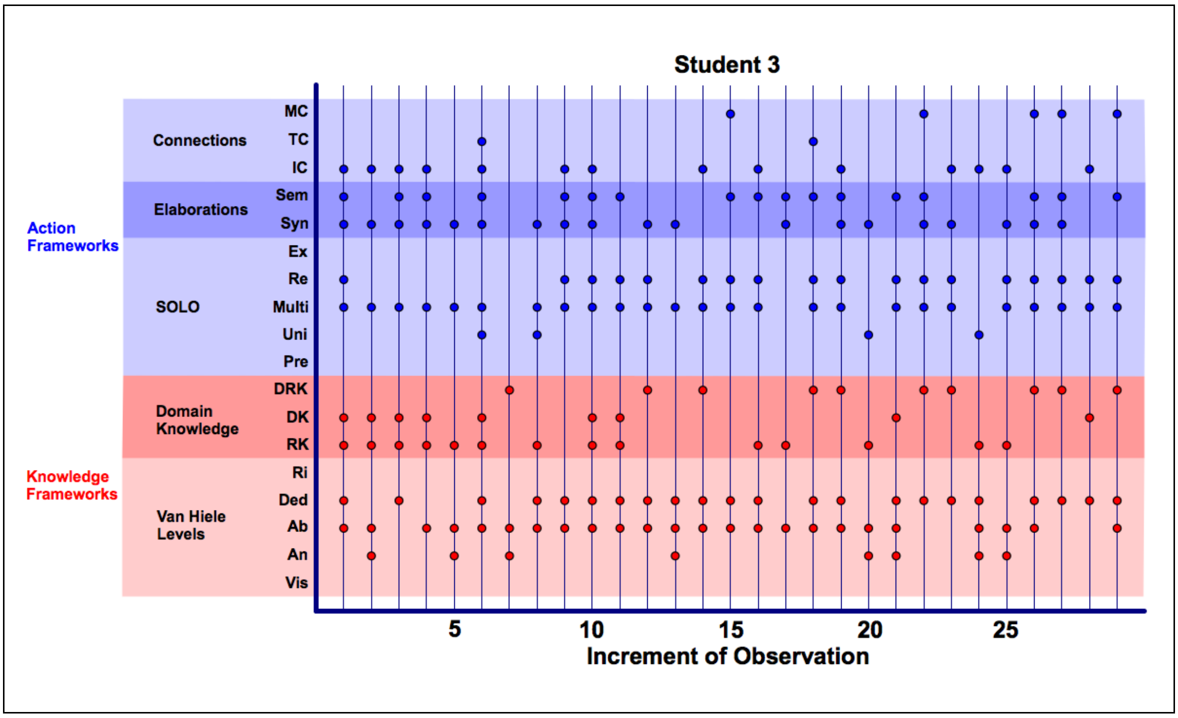
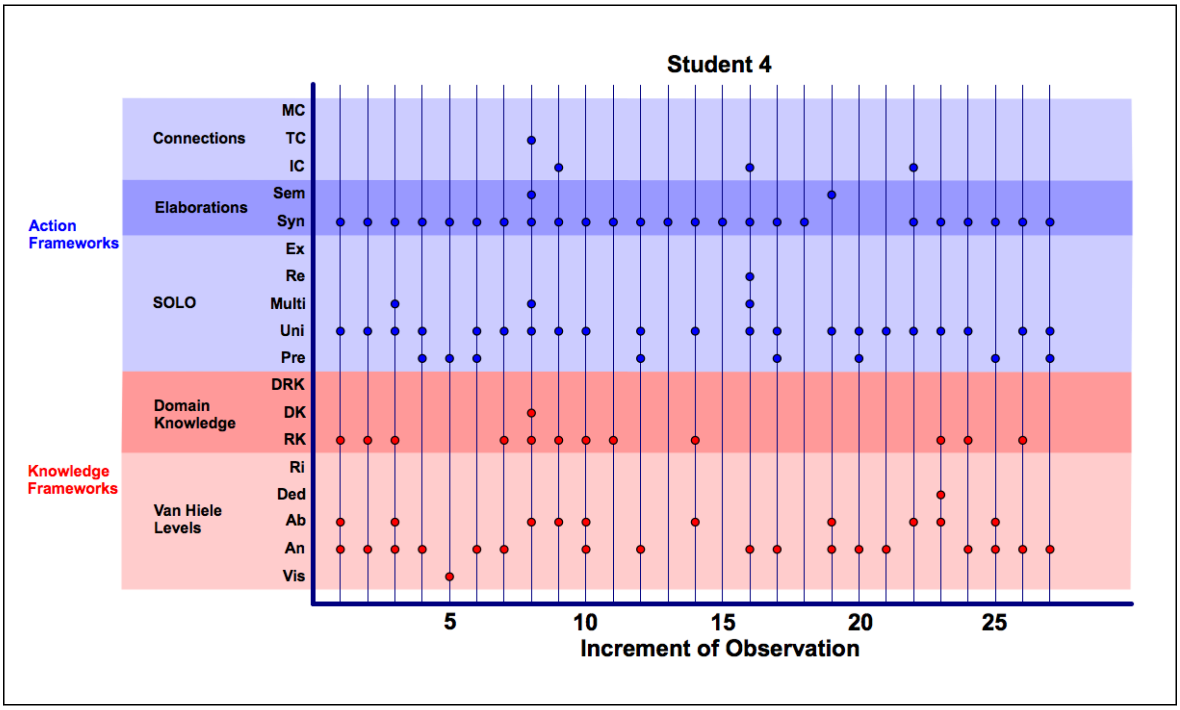


Figure 6. Student 3 Results



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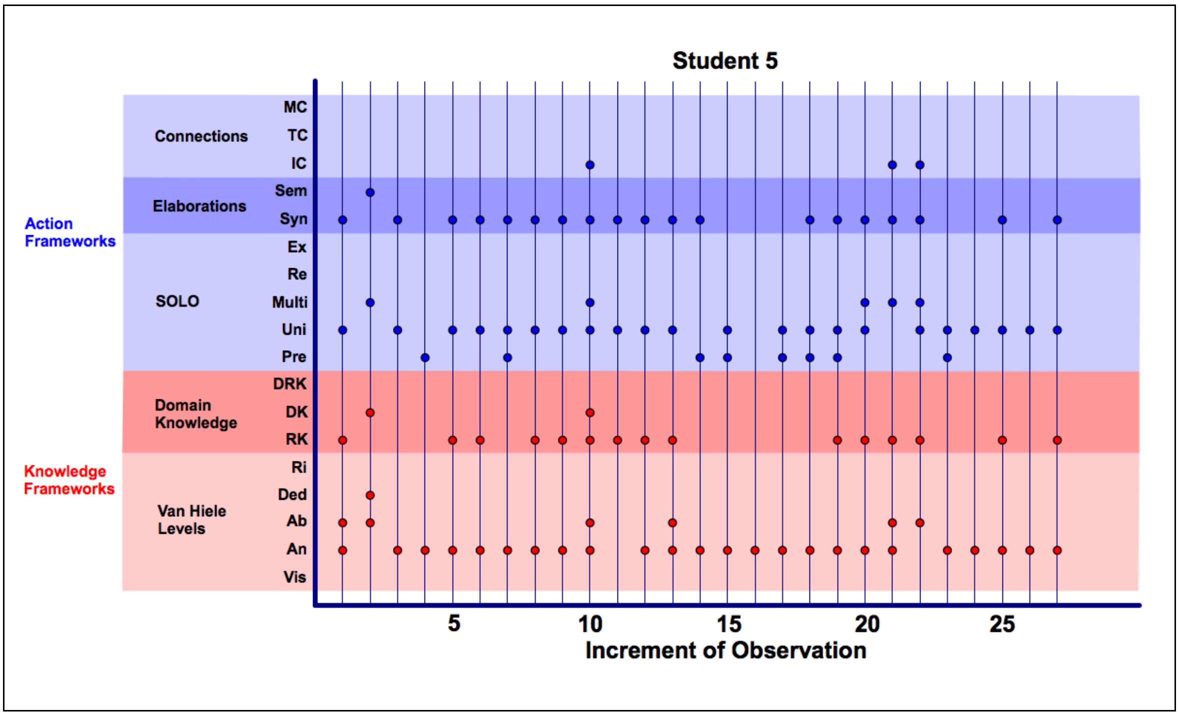

Figure 8. Student 5 Results

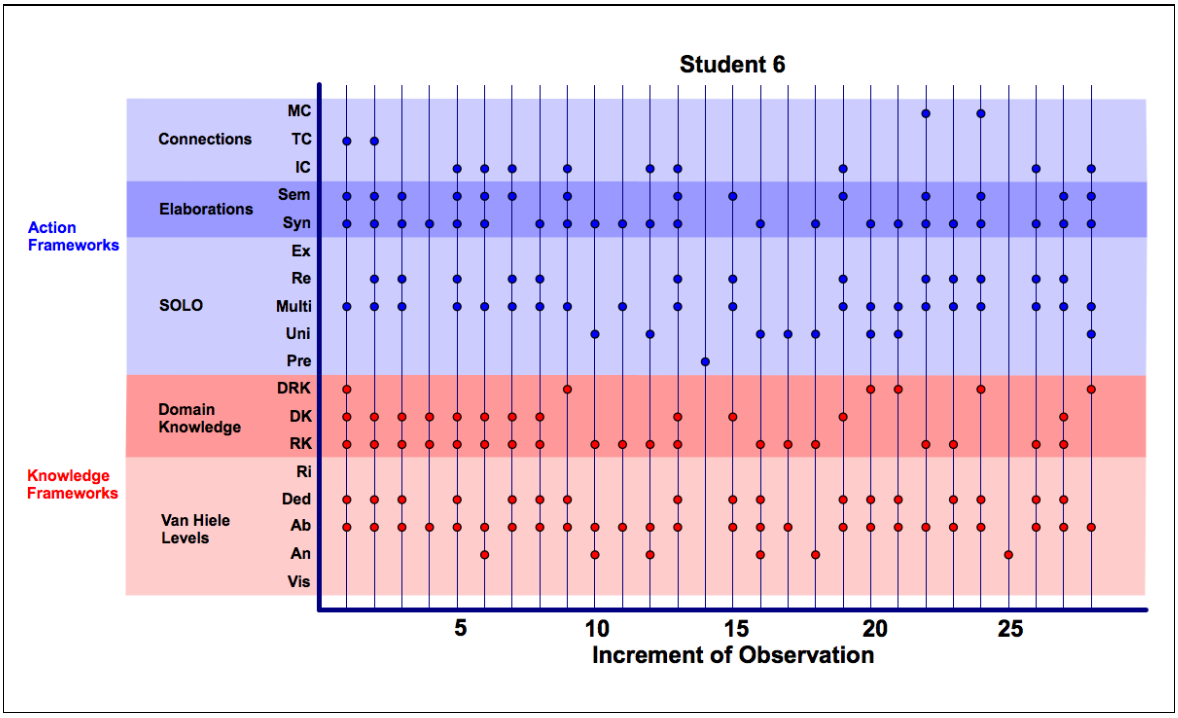

Figure 9. Student 6 Results

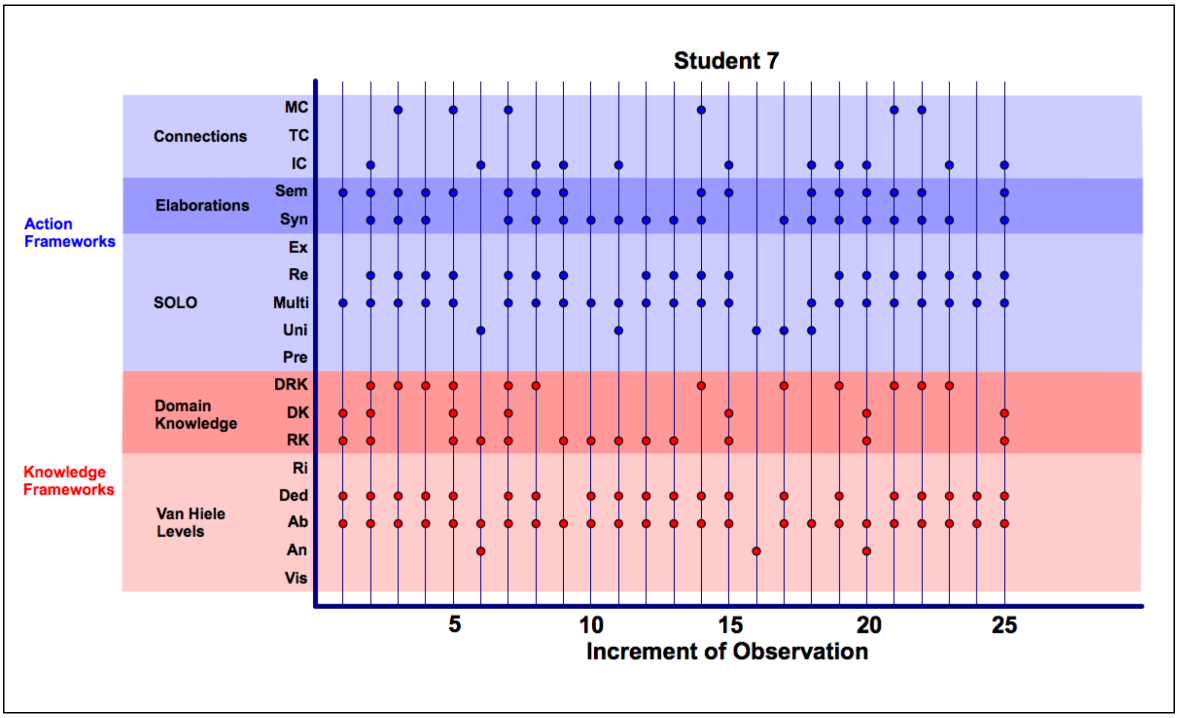

Figure 10. Student 7 Results